To Be Or Not Be

Next time you're at a dinner party, pick up a paper napkin, crumple without tearing, flatten energetically, and replace anywhere within its original outline. There will always be at least one point of the napkin which returns to its original position. This is known as the Fixed Point Theorem and is a famous example of a class of mathematical results known as Existence Theorems. As a party trick though, it can fall a bit flat.

Existence theorems are concerned with proving the existence - or nonexistence - of a solution to a given problem. Today, such theorems are recognised as useful in their own right, and mathematics does not require these theorems to offer the slightest insight into finding such a solution if it exists, or providing reasons why such a solution doesn't exist, if it doesn't. Such tasks can be left to others. In a world where efficiency is considered a virtue, it would seem to make sense to establish whether or not a solution exists before investing considerable resources trying to find it.

Outside mathematics, it generally comes as a bit of a shock to learn of questions without answers, problems for which no solutions exist. The knee-jerk reaction is usually surprise and rebuke – surely there's an answer to every problem, if one cares to look hard enough? Then a resigned, so what? Some mathematicians find remembering the day of the week tricky, never mind world-challenging problems like an effective anti-wrinkle cream, the correct shirt-tie combination or world peace. All the same, it is easy to see there are simple problems without solution.

Find a number which is greater than three and less than two. This is a simple to understand, perfectly well formulated problem in finding a number which simultaneously satisfies two constraints. It is also quite obviously a problem for which no solution exists. Perhaps a little too obviously – an ad hoc problem concocted to support a dubious viewpoint? Very well then, something stronger.

Find a number which, when I multiply it by itself and add the result to six, I arrive at five times the original number. A few guesses quickly unearth, not one, but two perfectly acceptable solutions – the numbers two and three.

Super. Now find a number which, when I multiply it by itself and add the result to *seven*, I arrive at five times the original number. You might approach this problem with a little more caution. It has no solution. Despite being virtually identical to the previous puzzle, you will not find such a number.

At least, that's what mathematicians would have said six hundred years ago. Today we say, "Solutions do exist, but they're not real". So what's going on here? Is mathematics just a word game after all? Can solutions exist today and not yesterday? Did the ancients get it wrong?

Mathematicians have learnt the hard way that the nonexistence of solutions to a problem, for which we have sound intuitive reasons to expect solutions, is the first inkling that something strange and interesting is lurking beneath the mathematical surface.

In this case, the nonexistence of solutions to the equation $x^2 - 5x + 7 = 0$ is a clue that the number system which we were trawling for solutions - and which we assumed “complete” - is in fact just a subset of a much more extensive and significant system known today as the set of complex numbers.

In this way, what might appear as a problem without solution, turns into a problem for which solutions do exist when set in a more abstract (albeit less intuitive) perspective. The challenge then is to find and solve further problems in this perspective and interpret their solution in terms of real world intuition.

The Greeks were the first to stumble onto a problem without solution, and subsequently realise the fact. Sound intuitive reasons led them to expect that the equation $p^2 = 2q^2$ had whole number solutions p and q, and that it was merely a matter of applying sufficient ingenuity and labour to uncover them.

p	q	p^2-$2q^2$
3	2	1
17	12	1
99	70	1
577	408	1
3363	2378	1
19601	13860	1
114243	80782	1
665857	470832	1
3880899	2744210	1
22619537	15994428	1

Many heroic attempts led to a host of values almost satisfying the equation, but no one could find an exact solution. Then, in a desperate bid to precipitate a break-through, somebody inadvertently proved that no such whole number solutions exist. This was totally unexpected and something the mathematical world was a long time accepting.

Another existence problem (for which the jury is still out) is whether every even number larger than two can be written as the sum of just two prime numbers:

4 = 2 + 2	12 = 5 + 7	20 = 3 + 17	28 = 5 + 23
6 = 3 + 3	14 = 3 + 11	22 = 3 + 19	30 = 7 + 23
8 = 3 + 5	16 = 3 + 13	24 = 5 + 19	32 = 3 + 29
10 = 3 + 7	18 = 5 + 13	26 = 3 + 23	……

Every even number tested to date is expressible as the sum of two primes. I won’t write out the entire list because it goes up to 400,000,000,000,000 and requires a sheet of paper the size of the Moon. But nobody knows whether *every* even number can be written as the sum of two primes. A single counter-example would suffice to disprove the assertion. This simple existence problem, now known as the Goldbach conjecture and which, according to G H Hardy "…any fool could have guessed …", nevertheless remains unsolved after more than 250 years. A prize of $1,000,000 and world fame awaits anyone who can settle the issue.

On the subject of prime numbers let's mention another problem, for which a solution does exist. It is quite easy to prove that while prime numbers get rarer with size, they never actually just stop. There is always another bigger prime further up the number line. A natural question then is, how much further? A little surprisingly, it turns out two successive primes can be separated by as many nonprimes as you care to request. In other words, there exist arbitrarily long strings of consecutive nonprimes somewhere up the number line. We can even say where.

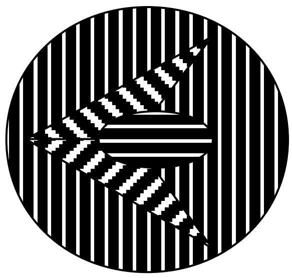
Let's start small and prove the existence of say a string of 100 consecutive nonprimes. I can tell you straight away where you can find one such string - starting at 2 + (101 x 100 x 99 x 98 x 97 x …. 4 x 3 x 2) . Proof:

2 + 101 x 100 x 99 x 98 x …. x 4 x 3 x 2 is divisible by 2
3 + 101 x 100 x 99 x 98 x …. x 4 x 3 x 2 is divisible by 3
….
101 + 101 x 100 x 99 x 98 x …. x 4 x 3 x 2 is divisible by 101

Large though these numbers are, they do form 100 consecutive numbers each of which is nonprime (divisible by a whole number other than 1 or itself). Clearly, we could continue in a similar way to find a string of a thousand, or a million, or a googol … consecutive nonprimes.

In a spirit typical of existence theorems, this only establishes the existence of one particular string of 100 consecutive nonprimes. It makes no comment on whether other strings of 100 consecutive nonprimes exist or even whether this is the first such string encountered up the number line.

Another famous existence problem concerns maps. Known as the four colour theorem, this problem was first posed in 1852. Does there exist a map which requires more than four colours to clearly distinguish the different regions? The diagram shows an example where four colours are required, but can more complicated maps be devised requiring five or more?



The problem was only solved in 1976 by Appel and Haken: No map exists requiring five or more colours. The proof consists of testing a huge (but not infinite) number of reduced networks. But many mathematicians are uneasy with the proof since the amount of analysis is so large that, to date, only a computer has been able to complete the calculations.

Fermat's Last Theorem is probably the most famous existence problem. Do there exist whole numbers a,b,c,n satisfying $a^n + b^n = c^n$ for $n > 2$?

For n=2 the problem reduces to Pythagoras' Theorem for a right angled triangle $a^2 + b^2 = c^2$, for which an infinite number of whole number solutions, or Pythagorian triples, are known (a,b,c) = (3,4,5), (5,12,13), (8,15,17), … together with all whole number multiples of these.

But do there exist any whole number triples when n = 3, 4, 5, …? Fermat himself said he had a proof that none exist, but never produced it. Since then, many famous mathematicians have tried to prove his theorem or find counter-examples. Nobody has ever found any, nor will they, for in 1993 Andrew Wiles stunned the mathematical world by proving their nonexistence.

Now all this is fairly abstract stuff, so maybe existence theory only has a significance in the ethereal world of mathematics and disappears like the morning mist in the glare of the real world? Comforting though such philosophies may seem at first, they turn out delusional.

Arguably, existence theory never had greater real world significance than in 1941 when Britain was losing the Second World War, while America remained aloof and prospering from the European carnage. Following a Polish breakthrough using mathematicians to decode Enigma ciphers, the British set up a code breaking unit at Bletchley Park. Among a small number of mathematicians studying the Enigma problem, was a gifted and socially awkward postgraduate student called Alan Turing.

Turing reasoned that, rather than look for the solution to an encryption problem, it might be more efficient to eliminate non-existent solutions. In searching for impossible solutions, huge side branches of impossible solutions could also be eliminated. Nevertheless the task remained monstrously huge, but it was Turing's genius to realise, in addition, that this approach could be mechanised.

By using an electrically operated switch called a relay (a device which, to the dismay of his teachers, Turing had toyed with endlessly at school), a machine could be built (the forerunner of today's programmable computer) which could systematically eliminate all impossible solutions, stopping only at the correct solution. Far from being a dubious mathematical abstraction, the non-existence of solutions, had suddenly become "as real as it gets". After the war, it was British government's turn stand aloof, while Alan Turing was socially hounded, until tragically solving his own existence problem by swallowing cyanide.

More than two hundred years earlier, Newton had solved the so-called two body problem of dynamics: How would two bodies, attracting each other gravitationally across empty space, move for the rest of time? He first formulated and then solved the mathematical equations of motion. In the case of the Earth-Sun system, Newton's solution has the Earth following a slightly elliptical orbit around the Sun, maintaining for all time a certain maximum and minimum distance from the Sun. Such stable solutions constitute the norm, filling the phase space of all possible solutions.

The situation for the three-body problem is entirely different. The equations of motion are no more difficult to formulate but neither Newton, nor anyone since, has been able to solve them. This would be just a mathematical inconvenience were it not for the awkward fact that the universe consists of manifestly more than two significant gravitational bodies, something like 10,000,000,000,000,000,000,000 in fact.

The full horror of the situation arises already in the three body problem. Under the introduction of a third body, the continuum of stable orbits, which filled the phase space of the two body problem, disintegrates into a small and highly disconnected remnant of stable orbits. In other words, a stable elliptical orbit of the two-body problem has little probability of surviving the introduction of a third body. Those which do survive find themselves surrounded by clouds of unstable orbits which can wander arbitrarily close to any point in phase space – including collisions with the Sun or ejection from the Solar System.

Moreover this general phenomenon, which has become known as Chaos Theory, is not confined to the n-body problem, but appears to be the generic behaviour of all interacting systems evolving in time. Today, the existence theory of stable and unstable solutions forms a huge area of mathematical research, with implications for long term stability of the solar system and other planetary systems, earth-asteroid collisions, galactic evolution, weather prediction, climate stability, population dynamics, species extinction, stability of financial markets, global economy and world peace.