Why Things Go Bump In The Night

Why will a frightened dog suddenly attack, or just as suddenly, cower? Why can an excited crowd become a riot or a loyal cabinet desert? Why do stock markets crash, world orders crumble, wars flare, earths quake, volcanoes erupt, lands slide, ice ages descend, magnetic poles flip, fault lines slip, ocean currents veer, climates jump, rain precipitate, snow crystallize, bridges fail, girders buckle, walls crack, floorboards creak, metals fatigue, plastic bottles ping, inspirations flash, illusions snap, images return, faces re-cognise, memories resurrect, forgotten words rebound? These are real world phenomena, familiar for their abruptness as their inexplicability.

The well-worn response is of course, that's just the way it is. Its neither necessary nor helpful to wonder why. That philosophically, no doubt, its even meaningless to ask. And indeed, that may be the case. Perhaps such problems are beyond human comprehension. But it was once considered pointless to ask why things fall. Nobody ever saw an apple fall up, but it took insight and courage to question why an apple should fall down. Familiarity and understanding are both useful attributes, but usually found on opposite sides of the coin.

Science does in fact hold short of asking the *why* question, contenting itself more with *how* things happen. How to analyse, how to predict and how to control things happening. But science is a product of human beings and humans are susceptible to social norms and customs. It takes time and not a little discomfort for science to abandon a cherished view, despite the change embracing image it likes to project. So when a new theory comes along, a theory which attempts to answer questions science has tended to avoid, and nonscience courted, controversy is bound to erupt.

Catastrophe Theory is the name that has come to label mathematical investigations into how some systems, whose behaviour is normally characterised by long periods of equilibrium and steady behaviour, can jump abruptly to another state. Introduced in the 1960's by the French mathematician René Thom, it suffered a troubled childhood, alternately castigated by the mathematical, scientific and nonscientific communities alike.

However, what matters in the long haul is whether an idea is useful, not the degree of emotion it generates. In this article we demonstrate the simpler ideas of catastrophe theory with a clear and careful analysis of a standard problem in mechanics. Here there is little room for doubt, either in the analysis or its conclusions, nor for philosophical mystification about the implications.

In the observable world our quest for knowledge and understanding is led by predictability. We feel happier when things which we expect to happen, happen. At the measurable level, we start out by looking for patterns of equilibrium and steady state. If a system is in equilibrium, and nothing disturbs it, we expect it to stay in equilibrium. If it moves away from equilibrium we hypothesise that some agent must have acted. These (in essence) are Newton's laws of motion and three hundred years of application to a multitude of diverse problems has resulted in a substantial body of understanding of motion and evolving systems in general. The analysis of systems in and near equilibrium, is fundamental to this study, for here we can apply all the tools of calculus to quantitatively determine how systems evolve in time.

For a very large class of important problems known as conservative systems, Newton's laws for the evolution of the system, can be recast as equations of motion for a hypothetical "system" particle moving in response to a potential energy U, a continuous function of a position variable ¯. Solving Newton's equations amounts to finding ¯(t), the functional dependence of ¯ on time t. This approach readily generalises for systems requiring more than one position variable ¯.

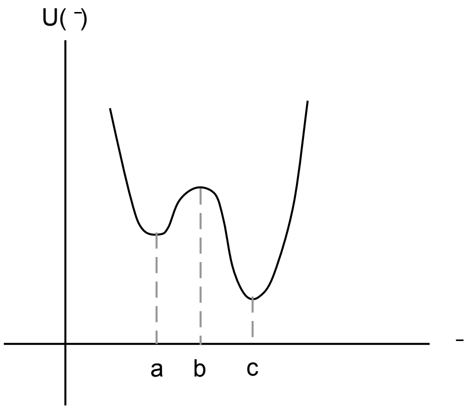
The force acting on the system particle is obtained from U(¯) as minus the rate of change of U with ¯, denoted by -dU/d¯. Where the graph of U against ¯ has zero gradient, that is where dU/d¯ = 0, zero force acts on the particle, and a particle initially at rest at this point will remain here for all subsequent time. Such points, $\theta = \theta_{eq}$, are known as stationary or equilibrium points. They specify states where system equilibrium is possible.

For example, consider a potential energy $U(\theta)$ whose gradient is specified by

$$\frac{dU}{d\theta} \propto (\theta - a)(\theta - b)(\theta - c) \qquad \text{where} \quad 0 < a < b < c$$

This system clearly has three equilibrium points corresponding to the values θ_{eq} = a,b,c. However the character of these equilibrium points, and the resulting behaviour of the system near such states, is quite different. This is most clearly demonstrated by sketching the potential energy U against θ (although for this somewhat contrived example, it is also possible to continue algebraically).

We know the gradient of the curve must become zero at the three well-separated points θ = a,b,c. We also know that $U(\theta)$ must contain a θ^4 term, which dominates all other terms for large (positive and negative) values of θ. Together with the knowledge that U is a continuous function of θ, this is sufficient to sketch its curve.



Actually, some additional information on the values of a,b,c (and a fair bit of calculation) is required to decide if the local minimum at θ = a is higher or lower than that at θ = c. All we know is they are well separated by a local maximum at θ = b.

With force given by $-dU/d\theta$, we see for values of θ near a <u>minimum</u> of U, the force opposes a displacement from the minimum, whereas for values of θ near a <u>maximum</u> of U, the force reinforces a displacement from the maximum. Hence, the minima and maxima of $U(\theta)$, correspond to stable and unstable stationary states of the system, respectively. Particles tend to move off hilltops and gather in depressions. This conforms with our intuition for the motion of a particle on a smooth surface under gravity, the original problem from which this construction derives.

Because the resulting behaviour of a system at or near a stable stationary point is completely predictable, much time and effort in science is devoted to formulating the appropriate potential energy function of a system and locating its minima. Even for non-conservative systems, where energy can feed in and out of the system, provided the energy flux is “small”, the system remains in a neighbourhood of the stable equilibrium, exhibiting steady long term behaviour.

If only this were all, our world would be a calmer and more predictable place. However, although we live in a world in which mathematics appears to describe nature “unreasonably well”, mathematical models are always incomplete to some degree. Parameters of a mathematical model, such as the values a,b,c above, which for mathematical convenience we considered constant, will generally be susceptible to outside influences. As a result, we should expect their value to drift slowly, rather than remain rigidly fixed. This drift can have significant consequences for the stationary points of a system.

To demonstrate this, we to return to above system and repeat the analysis, this time explicitly acknowledging that the parameters a,b,c are slowly varying functions of (say) two new *control variables* x,y. We will find this gives rise to a situation where two stationary points can “collide”, resulting in a change of stability for each, and potentially, an abrupt change in the stationary state of the system – a catastrophe.

We set $a = \alpha + f(x,y)$ $\qquad b = \beta + g(x,y)$ $\qquad c = \gamma + h(x,y)$

where ✓, ✓, Γ are constants, f,g,h are slowly varying continuous functions of x and y, and f(0,0) = g(0,0) = h(0,0) = 0

With this expanded role for a,b,c the above analysis is reproduced simply by setting the control variables to x = y = 0. Stationary points of U(¯) arise at the same (but renamed) values ¯$_{eq}$ = ✓, ✓, Γ and with the same stability characteristics as before.

Regarding now (x,y) as a single point in the (x,y) plane - the control plane - we can plot the equilibrium positions ¯$_{eq}$ as three dots at heights ✓, ✓, Γ above the origin of the control plane (x,y)=(0,0), Fig 2a.

If we knew the functional dependence of f,g,h on the control variables, we could now proceed to calculate three stationary values ¯$_{eq}$ for another point (x,y) near (0,0) in the control plane, (where shading indicates their stability character (vertical = stable, horizontal = unstable). Proceeding outwards in this way from the origin of the control plane, we generate sheets of stationary points above the control plane, the height of each point of each sheet being a continuous function of the control point (x,y) in the control plane (Fig 2a).

We are able to propose this construction - despite a lack of knowledge about the functional forms f,g,h - by the virtue of one critical and powerful assumption. Namely that the f,g,h are all continuous functions of the control variables x and y. This has the consequence that the height of each equilibrium point ¯$_{eq}$ is then a continuous function of its control point. A nearby control point generates a nearby stationary point. The sheets, while varying in height above the control plane, are continuous surfaces with neither holes nor tears (Fig 2a).

However, while the sheets must be remain continuous, we cannot continue outwards indefinitely before something interesting happens, namely, the sheets start to intersect or coalesce with each other. Recall that intersection of the sheets signals collision of stationary states, that is, catastrophes.

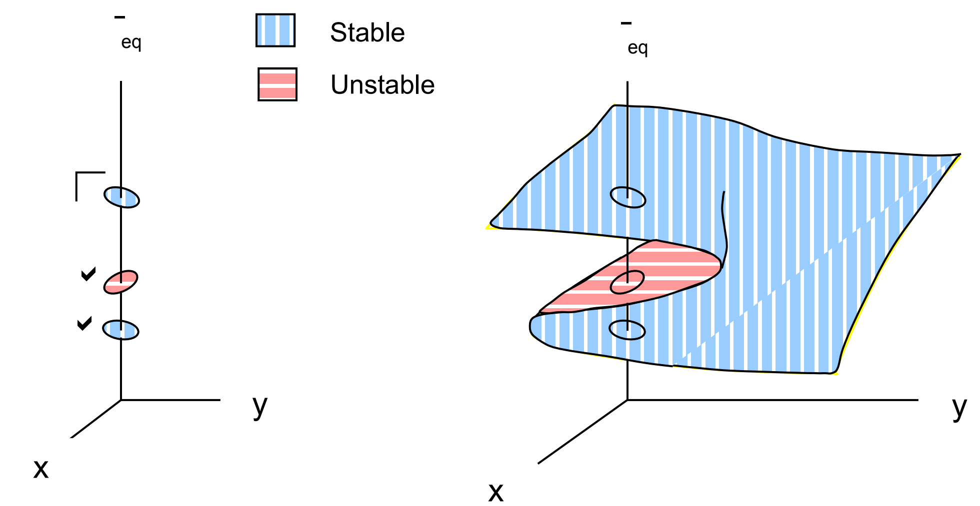
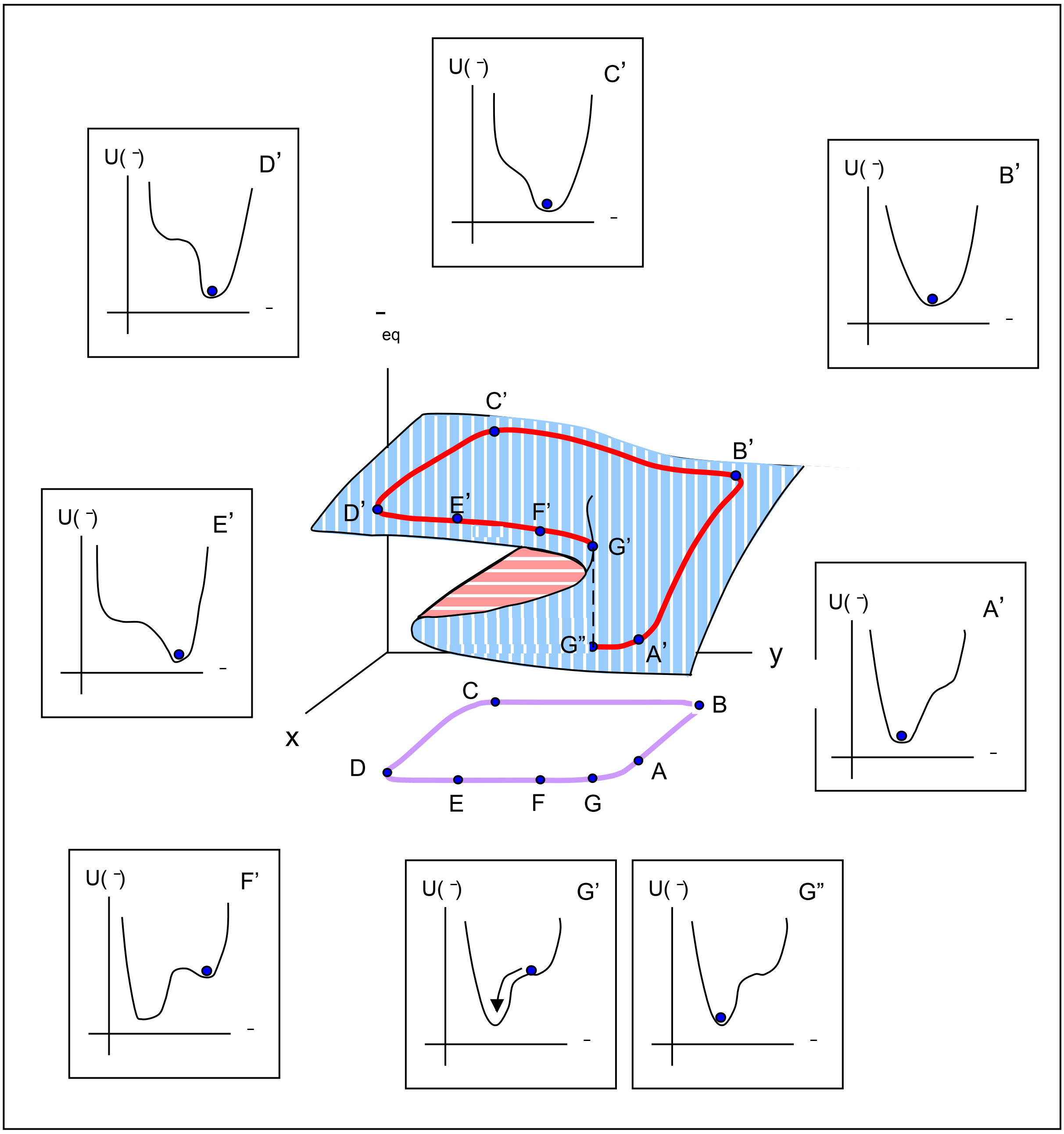


Fig 2a Generating sheets of stable and unstable stationary points for the potential energy function U(¯). The height of each stationary point ¯$_{eq}$ is a continuous function of the control variables (x,y).

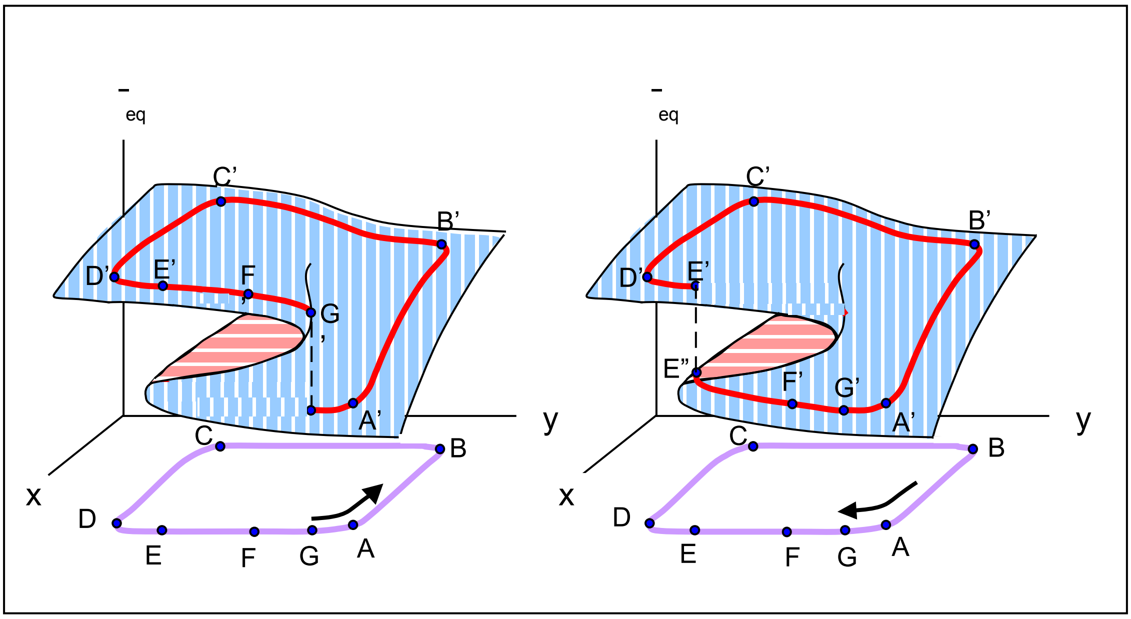
Fig 2b The cusp - one of the simplest catastrophes. Generating the sheets algebraically for realistic potentials is difficult. Topology reveals only seven generically distinct types of catastrophe.



Before looking at catastrophes in general, we consider the behaviour of a mechanical system near a cusp catastrophe, one of simpler possible types. Fig 3 shows the variation of a potential energy function U(¯) and its stationary states ¯$_{eq}$, for various points of a closed circuit ABCDEFGG’ in control space.

The stable stationary state of the mechanical system tracks continuously over the equilibrium surface starting at A’ through B’, C’, D’, E’, F’ until it approaches G’. At G’ the occupied stationary state destabilises, and the mechanical system suffers an abrupt transition as G” becomes the occupied stable stationary state of the system, before tracking continuously once more from G” to A’.

The mechanical system exhibits quite different behaviour depending on the sense in which the control circuit is traversed (Fig 4). Following the closed path AGFEDCBA, the stationary state of the mechanical system tracks continuously over the equilibrium surface from A’ through G’, F’, until it approaches E”. At E” the occupied stationary state destabilises, and the mechanical system suffers an abrupt transition as E’ becomes the occupied stable stationary state, before tracking continuously once more through D’, C’, B’ and on to A’.



The above process demonstrates the phenomenon of hysteresis in a mechanical system around a catastrophe. Circuiting a catastrophe in opposite directions gives rise to distinctly different responses in the mechanical system.

Now, just when things get interesting, we run into a technical difficulty. Generally, we can't solve the necessary algebraic equations to plot the equilibrium sheets and their intersections. Further investigation requires one of two divergent methods, and it is here once more, that controversy ensues.

One approach is to calculate the equilibrium sheets for as many simple systems as possible and hope the results are representative of the general case. Unfortunately, a few abandoned attempts with even the simplest of systems are enough to convince most people of the algebraic obstacles of such an undertaking. Even in the example above, carefully engineered to permit the complete calculation and characterisation of stationary points, we quickly resorted to powerful doses of continuity to ease the analysis. The complexity becomes monstrous when we start to consider realistic potentials with more that one position variable ¯, and control functions with more than two control variables x,y

It is here, just like the cavalry, that topology comes to the rescue. Topology says, disband the algebraic battalions. Retaining only the assumption of continuity, attack the problem from the blind side of the hill. Assuming only continuity, how many generically distinct types of surface intersection is it possible to construct anyway? And then, in how many generically distinct ways can each such intersection arrange itself above the control plane? Figure out these combinations and there are no other options. Any catastrophe for a real world system must then described by one of these.

At first we need only the landscape of our real world to visualise such surfaces - hills, depressions, valleys, peaks, mountain ridges, passes, cols and overhangs. As dimensions increase, our visual intuition fails, but topologists continue effortlessly, collecting specimens of the topological landscape, lovingly categorising each species of catastrophe with enigmatic names like fold, cusp, swallowtail, butterfly, hyperbolic umbilic, elliptic umbilic and parabolic umbilic,

The last mentioned is a highly illusive species, inhabiting a world of four control variables and two position variables. Thankfully, even topologists start to struggle in six dimensions so that only seven distinct species of catastrophe are known.

Now while it may be mathematically natural to introduce control variables, enabling the catastrophe to be “unfolded” in the additional dimensions, we need to be aware that in the real world, control variables are likely to be difficult measure, enumerate or even identify. It is on this point that much of the controversy with catastrophe theory rages. This is a pity, because the difficulty in identifying control variables is entirely irrelevant to the qualitative conclusions of the mathematical analysis – how catastrophes arise and what they look like.

Generically, stationary points will collide and catastrophes will happen, it is too contrived mathematically and naïve physically to suppose otherwise. In a complex interdependent world, it is hard to fault the topological approach of emphasising generic or “structurally stable” systems. Knowledge of the precise functional dependence on the control variables merely supplies the where and when. Equally though, no one should deny that, one day, it might prove useful to know the where and when for certain systems.

The evolution of a system away from a newly destabilised equilibrium, while sudden, is not discontinuous, does not defy the speed of light, the conservation of energy or any other law of physics. It is not even contentious, being perfectly well specified by Newton’s laws of motion for a particle moving between unstable and stable equilibrium points. But when a system’s equilibrium destabilises, its transition to the nearest accessible stable equilibrium is rapid, relative to its previous steady behaviour, and for that reason deserves to be called abrupt. And if the system in question concerns a planet’s climate or a living species’ continued existence, then perhaps the term catastrophe is fairly apt.

Nobody argues for catastrophe theory as the definitive explanation for all the abrupt phenomena mentioned earlier. But if the simplest continuous systems of Newtonian mechanics exhibit catastrophes, we should certainly expect real world systems, with their greater connectivity and interdependency, to be at least as complex and intriguing.