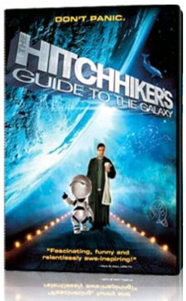
Infinity and Beyond (scary version)

The Hitchhikers Guide to the Galaxy begins,



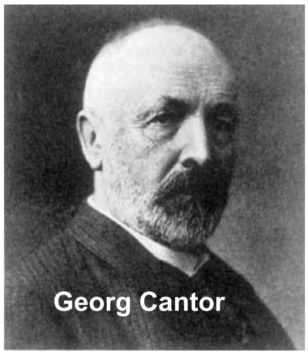
"Space is big. You just won't believe how vastly, hugely, mind-bogglingly big it is."

The key word here is *believe*. Observing the existence of 10^{22} stars across 10^{23} km of space, is one thing. Believing the implications of such numbers is something else.


Hubble Deep Space image

But if you think space is big, you should book a round trip to Infinity. There, mathematical peaks soar indefinitely over your head, bottomless ravines gape beneath your feet. Monstrous creatures lurk in ambush across a land where none of the familiar rules of mathematical engagement apply. Few mathematicians go near the place and many refuse even to acknowledge its existence. And for good reason.

Georg Cantor, the first pioneering mathematician to explore infinity - and return to tell the tale - went quietly mad, dying a lonely death in a German mental asylum. Anyone foolish enough to venture there today risks forfeiting academic funding and medical cover. Yet, none of the main results are difficult to understand, just difficult to believe. That's the scary thing. Infinity is understandable, just not believable.


Georg Cantor

As a breed, philosophers are generally less reticent expressing definitive opinions on infinity and mathematics.


Georg Hegel

I cannot help it, - in spite of myself, infinity torments me. Alfred de Musset.

Man is equally incapable of seeing the nothingness from which he emerges and the infinity in which he is engulfed. Blaise Pascal

Would they but pay some attention to philosophy, they must see immediately that there can be precisely seven planets, no more, no less. Georg Hegel

Mathematics has the completely false reputation of yielding infallible conclusions. Its infallibility is nothing but identity. Two times two is not four, but it is just two times two, and that is what we call four for short. But four is nothing new at all. Johann von Goethe.

Eternity is a very long time, especially towards the end. Woody Allen

Before embarking on a hazardous journey it’s prudent to warm up with a few practice exercises. These questions won’t deal with infinity itself, although they will involve some pretty big numbers - and some unsettling conclusions.

There are some 50,000 grains in this 1 Kg bag of rice. If you place 1 grain on the first square of a chess board, 2 on the second, 4 on the third, 8 on the fourth, and so on, doubling the amount on each successive square, quite surprisingly you run out by the 16th square.

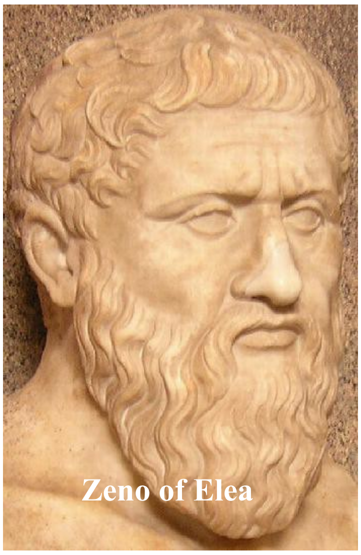
Ordering a one tonne pallet just to be on the safe side (there’s always Monday curry) you are slightly piqued to run out again, this time at square 26. Still, that’s almost halfway down, can’t be far now, so you get on the phone and order a 1000 tonne shipment. That, it turns out, just gets you to square 36.

You’ve just blown the school’s entire food budget for a year, not to mention, triggered panic buying on the London commodities market. But you’ve studied PPE and you admire Lady Thatcher, so you’re not for turning back now. You take the decision and send out for whatever it takes to finish the job. To the sound of lorries rumbling down Palace Street, you sit down for the first time and do the calculation. Let’s see,

$1 + 2 + 4 + 8 + 16 + 32 + \ldots$

$$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + \ldots + 2^{63} = 2^{64} - 1 \approx 10^{19}$$

Well, 10^{19} grains doesn’t sound too bad, at least you can write it on the back of a postage stamp. So what’s that in weight? At 5×10^4 grains to the Kg that weighs in at about 2×10^{14} Kg. The world’s human population is nudging 7×10^9, so that’s about 30 tonnes of rice per person. Sufficient to feed the entire world for 100 years. Or if you prefer, a rice mountain 300 times the size of Mount Everest sitting on Green Court. Time to make another phone call.



10^{19} grains might seem as good as infinity in the real world, so do we ever need to worry about an actual infinity? Are there real world problems which require infinity’s existence? It’s a slightly circular question since you can always come back and say you don’t consider *such and such* a real world question, but yes there are.

Zeno’s famous paradox of the tortoise and the hare, for example. Let’s use some actual numbers, you can always mystify the problem later.

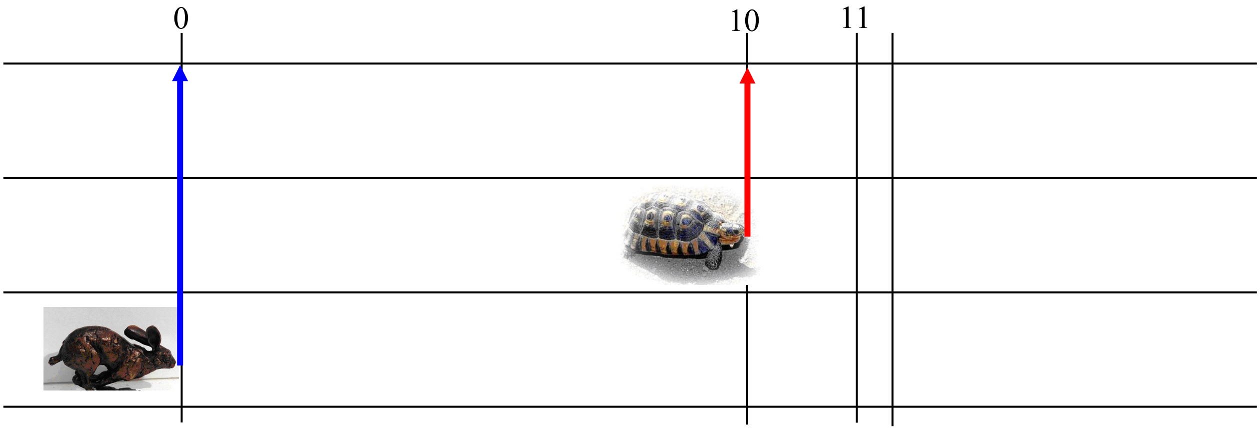
The hare runs at 10m/s, the tortoise at 1m/s. The hare graciously gives the tortoise a 10m head start.

After 1 second the hare has caught up with where the tortoise *was*, but the tortoise has moved on another 1m in that time.

After a further 1/10 second the hare has caught up with where the tortoise *was*, but the tortoise has moved on another 1/10 m in that time.

After a further 1/100 second the hare has caught up with where the tortoise *was*, but the tortoise has moved on another 1/100 m in that time.

Clearly this *process* will continue for ever. So while the hare runs faster than the tortoise, it <u>never</u> completely catches up with the tortoise, there's always a little more distance to cover.



Now, you are missing the point of the paradox if you point out, quite correctly, that in $\frac{10}{9}$ seconds the hare and tortoise will be shoulder to shoulder at $\frac{100}{9}$ m.

The point of the paradox is, both arguments appear flawless, yet they reach different conclusions. Mathematics is in trouble if different routes lead to different conclusions, even if the simpler perspective yields a more "acceptable" answer. The fallacy must be found and resolved if mathematics is not to dissolve into a sea of contradictory half truths.

The fallacy occurs at the underlined word, <u>never</u>. Mathematicians are human beings (just) and humans are obliged to communicate via the vagaries of human language. Usually these ambiguities are fairly mild and obvious and easily avoided through careful definitions. However this is an instance where the subtly different interpretations of "never" result in a contradiction.

The first argument uses the word "never" to mean the hare cannot cover an infinitely many intervals of distance required to draw level with the tortoise. But while the total distance is formulated as the sum of an infinite number of finite amounts, the total itself need not be infinite in value. And in this case, is not infinite, but actually a very sensible finite value:

$$10 + 1 + \tfrac{1}{10} + \tfrac{1}{100} + \tfrac{1}{1000} + \cdots = 10.111... = \tfrac{100}{9}$$

Zeno’s paradox is only a paradox if you insist the sum of an infinite number of fractions has to be infinite.

Of course, the use of arguments involving infinity, when perfectly straight forward alternatives are available, could be argued a perfectly crazy thing to do. However, as is so often the case in mathematics, practice and familiarity with infinity in problems where simpler methods provide a consistency check, is but preparation for the day when handling infinity itself becomes unavoidable. So with that endorsement, let’s now consider infinity proper.

Here is a 2m long piece of wood. You can’t see it because it’s a mathematical piece of wood, a line segment of length 2. If I cut it into two equal parts, I get two identical line segments each of length 1. The total length of course remains unchanged at 2:

$$2 = 1 + 1$$

Now I can take one of the smaller parts and do the same thing again. The total remains unchanged at 2:

$$2 = 1 + \underbrace{\frac{1}{2} + \frac{1}{2}}$$

Now I can take one of the smaller parts and do the same thing again. The total remains unchanged at 2:

$$2 = 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}$$

Now I can take one of the smaller parts and do the same thing again. The total remains unchanged at 2:

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \underbrace{\frac{1}{8} + \frac{1}{8}}$$

And so on, and so on, and so on,

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + ...$$

where those three little dots stand for “and so on for ever”. Congratulations, you have just calculated the sum of an infinite number of fractions

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + \frac{1}{4096} + ... = 2$$

The result is only exactly 2 if you sum the <u>infinite</u> number of fractions.

There can be no sitting on the fence here, waiting to assess the philosophical fallout before accepting the result. The sum is necessarily 2. Any other conclusion, less than 2 or greater than 2, immediately leads to fundamental contradictions in elementary arithmetic.

With this discovery, and a little practice, an entirely new mathematical tool called “series expansion” becomes available. For example, applying the “split in half” process to the individual terms themselves, yields:

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + \frac{1}{4096} + \ldots$$

$$\begin{array}{cccc} \frac{1}{2} & & & \\ + & & & \\ \frac{1}{4} & \frac{1}{4} & & \\ + & + & & \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \\ + & + & + & \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ + & + & + & + \\ : & : & : & : \end{array}$$

$$2 = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \frac{7}{128} + \frac{8}{256} + \frac{9}{512} + \frac{10}{1024} + \frac{11}{2048} + \ldots$$

a less obvious, but equally interesting discovery about 2.

Of course, there’s nothing special about 2. We could divide throughout by 2

$$1 = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + \frac{6}{128} + \frac{7}{256} + \frac{8}{512} + \frac{9}{1024} + \frac{10}{2048} + \frac{11}{4096} + \ldots$$

and derive an equally useful expansion for 1. Armed with an expansion of 1, we can expand any number.

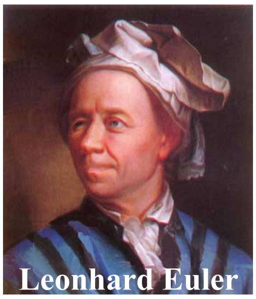
The modern perspective is actually the converse: Some infinite series do “consolidate” down to form the familiar whole numbers and fractions (collectively known as the rational numbers), but others (the vast majority in fact) do not. These latter series define an entirely distinct set of numbers called the irrationals, of which $\sqrt{2}$ and π are two famous examples.

$$\pi = \frac{4}{1} + \frac{4}{3} - \frac{4}{5} + \frac{4}{7} - \frac{4}{9} + \frac{4}{11} - \frac{4}{13} + \frac{4}{15} - \frac{4}{17} + \frac{4}{19} - \frac{4}{21} + \frac{4}{23} \ldots$$

$$\sqrt{2} = 2 - \frac{1}{2} - \frac{1}{16} - \frac{1}{64} - \frac{5}{1024} - \frac{7}{4096} - \frac{21}{32768} - \frac{33}{131072} \quad \ldots$$

Distressingly for them, the Greeks were the first to discover that whatever else $\sqrt{2}$ was, it wasn’t a fraction, but required an infinity of fractions for its representation. I won’t test your patience with the proof, but again, it is surprisingly straight forward. By consistently representing every number -

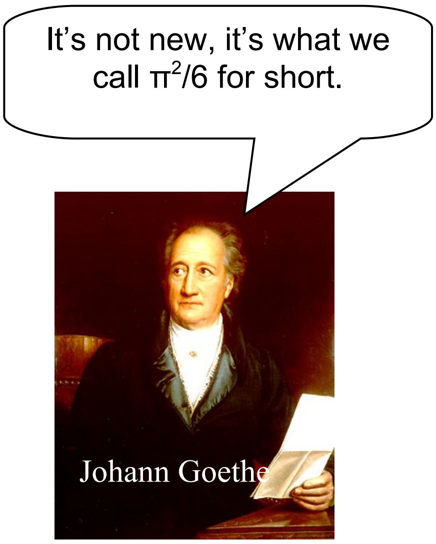
rational or irrational - as an infinite series, mathematics today confers equal status on all numbers, albeit at the cost of abandoning some naive perspectives.



Once admitted to the mathematical workshop, working with an infinite number of terms becomes commonplace. Infinite series turn out to be an Aladdin's cave of mathematical techniques and results. Leonhard Euler was acknowledged master of the technique, dismantling and re-assembling terms as a child might play with Lego bricks. In his hands, infinite series proved amazingly versatile, settling a vast range of problems and ushering in completely new areas of mathematics.

Euler was especially fond of expansions of the type

$$\frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\frac{1}{5^z}+\frac{1}{6^z}+\frac{1}{7^z}+\frac{1}{8^z}+\frac{1}{9^z}+\frac{1}{10^z}+\frac{1}{11^z}+\ldots=$$



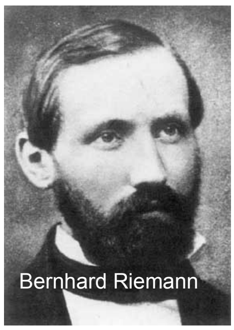
for special values of z. For the particular value z = 2, Euler was able to calculate the sum:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}+\frac{1}{11^2}+\ldots=\frac{\pi^2}{6}$$

a deep and unexpected connection between the natural numbers and the circle. The calculation for z=2 is now called the Basal problem, but Euler made extensive calculations for other values of z. But neither Euler nor anyone since, has succeeded in summing the series for z=3. A golden opportunity for a modern day Goethe, perhaps.

Clearly, the total value of the infinite series will depend on the value selected for z. But what if we now turn the question around and ask, for what value of z does the infinite series sum to a given value, say zero:

$$\frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\frac{1}{5^z}+\frac{1}{6^z}+\frac{1}{7^z}+\frac{1}{8^z}+\frac{1}{9^z}+\frac{1}{10^z}+\frac{1}{11^z}+\ldots=0$$



This is the famous Riemann equation. At the outset, it's not immediately obvious there is any value of z which causes the series to sum to zero. And indeed, there are no *real* values of z satisfying this equation. However if we allow z to wander off the real axis into the complex plane, rather surprisingly we find an infinite number of z values satisfying this equation.

This set of values (a little confusingly called the zero's of the Riemann equation) has some remarkable properties. For one thing the set is intimately connected with the set of prime numbers (it was a study of prime numbers which led to the formulation of this series in the first place).

For another, the solutions all appear to lie exactly on the same straight line in the complex plane, $x = ½$. Of course, any 2 points will always lie on some line. Any three points generally won't, so there's something special if 3 or more points lie on the same straight line. To find that not three, but the first 10^{13} zeros all lie exactly on the same straight line, as computer calculations have discovered, is suspicious to say the least, but to this day nobody knows if *all* the solutions lie on this line. A single exception would disprove RH.

The Riemann Hypothesis has become a kind of star-gate to another dimension of the mathematical universe, a portal through which totally unrelated fields of mathematics become strangely connected and unified. Intrepid explorers, tired of waiting for the train of rigorous proof to arrive, have already set off on foot to explore this land. So many strange and wonderful creatures have been discovered in this land (if it exists!), that many mathematicians will be devastated if RH proves false. The situation is so desperate, the Clay Institute of Cambridge, Massachusetts, is currently offering a prize of one million dollars to anyone who can settle the question one way or another.

Sooner or later, every school child learns to play the infinity game.

> *"What's the biggest number in the world?*
> *Megagoogalsqillion!*
>
> *Wrong, knucklehead!*
> *Megagoogalsqillion and one. Ha! Ha! Ha! You don't know anything!"*

This "and one" notion is, in fact, the present day basis for the construction of infinity. Consider the process of adding 1 repeatedly:

$$1 \xrightarrow{+1} 2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1}$$

Clearly, the process either leads to

(a) a finite number, or
(b) a non-finite number

If it leads a finite number, then we can continue the process - we know how to add 1 to any finite number. Eventually, we are forced to either

(a) conclude the process generates a non-finite value, or
(b) surrender the assumption that we can add 1 repeatedly

The non-finite value we elect to call infinity. You may harbour lingering doubts about the “repeatedly” aspect. If so, you need to re-examine every other repeated process which, until now, you have happily entertained. Such as adding 0 repeatedly (or equivalently, add 1 then subtract 1, repeatedly)

$$1 \xrightarrow{+0} 1 \xrightarrow{+0} 1 \xrightarrow{+0} 1 \xrightarrow{+0}$$

If that process is unsafe, adding zero repeatedly, will eventually fail to return the original value. A very drastic revision of arithmetic would then ensue.

We have reached certain conclusions about infinity:

(a) Infinity is more a process than a value
(b) We generated this infinity via the “add 1” process (there may be others)
(c) No number bigger than infinity is accessible by adding (◎ ▯ 1 ▮ ◎)

Historically, the “add 1” process led to this infinity being called the “countable infinity”, since adding 1 is essentially counting. The first letter of the Hebrew alphabet, aleph, has been assigned to denote this infinity $\aleph_0$ (the zero subscript hints of bigger infinities to come).

Now while $\aleph_0$ is very unintuitive, it is comfortingly remote, and so long as we just stick to addition and subtraction, this infinity remains remote. Yes, $\aleph_0$ is a monster, but monstrously far away. It could be worse. In fact, it soon will be.

Meanwhile, $\aleph_0$ itself has some very unintuitive properties. If you insist on comparing the number of terms in the two sequences,

1, 2, 3, 4, 5, 6, 7, 8, 9, … (the set of natural numbers)
1, 4, 9, 16, 25, 36, 49, … (the set of square numbers)

you could be forgiven for imagining the first set contains very many more members than the second set. After all, the first sequence contains every term of the second sequence - and a lot more besides. On the other hand, written as

1, 2, 3, 4, …

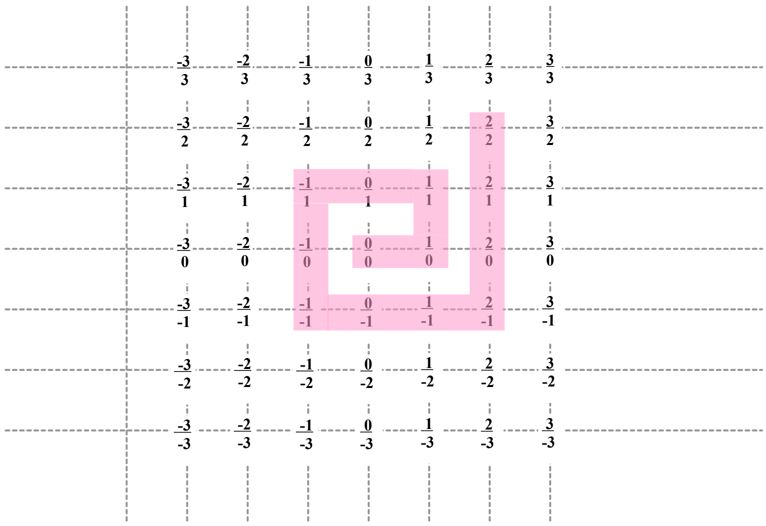
$1^2, 2^2, 3^2, 4^2, \ldots$

it becomes clear there is a one to one correspondence between the members of each sequence, the first sequence effectively counting off the second sequence. The only mathematically consistent conclusion is that the sets contain exactly the same number of terms as each other. We say they have the same *cardinality*, aleph-null. You can’t increase $\aleph_0$ by adding to it a finite or even an infinite amount.

Similarly, you could be forgiven for imagining there are vastly more fractions than whole numbers. After all, between any two consecutive whole numbers you can always find two fractions, and between these two fractions, another two fractions, and so on. But it turns out there are just as many fractions as whole numbers, no more, no less, as Hegel might say. The cardinality of the rationals is precisely aleph-null. Once more, this is disconcertingly easy to prove (and equally hard to believe).

Start by laying out the fractions on a rectangular grid. All fractions with same numerator are ordered vertically and all fractions with the same denominator are ordered horizontally. It is clear that this listing will feature all possible fractions somewhere (many times over in fact, but that does not matter).

Now trace a spiral path linking every fraction, moving out from the centre, and counting off the fractions in one to one correspondence with the integers. Since there is nothing to stop you doing this indefinitely, you have just proved the fractions have the same cardinality as the integers, aleph-null.



By now you are probably resigned to thinking that simply every transfinite quantity is aleph-null. Absolutely not so. The real numbers (the combined sets of rational and irrational numbers) turn out to be infinitely bigger than the set of fractions. Again, once seen, the proof is eerily simple.

Suppose the cardinality of the real numbers *was* aleph-null. Then, like the whole numbers and fractions, it would be possible to arrange them as a complete list, a tiny and very incomplete portion of which might be

…
…72930547923.09505783…
…83669484668.85798044…
…25736257943.58374456…
…14734724636.84457129…
…68333847979.99293886…
…32037388607.31528553…
…40832643913.23304004…
…59338284632.12648473…
…

We will not require this list be ordered in any particular way, just <u>complete</u>. Every real number must feature somewhere in the list.

Now, starting at the decimal point of some number, proceed outwards in both directions highlighting successive digits

…
…72930547**9**23.09505783…
…83669484**6**68.85798044…
…2573625794**3**.58374456…
…14734724636.**8**4457129…
…68333847979.9**9**293886…
…32037388607.31**5**28553…
…40832643913.233**0**4004…
…59338284632.1264**8**473…
…

Assemble the highlighted digits to form a new number, ….963**.**89508…

Finally, perform the digit shift 0→1, 1→2, … , 8→9, 9→0 to obtain

….074**.**90619…

Of what conceivable interest, I hear you ask, could anyone ever have in this number? Well, it's not on the list. You know, the list which we were very careful to stipulate was a <u>complete</u> list of all the real numbers.

For, wherever else this number is, it's clearly not in the portion depicted, differing by at least one digit from each of the numbers featured. But by the same token it isn't anywhere else in the list either, since by construction, it differs in at least one digit from every number listed.

The only consistent conclusion has to be that, contrary to our initial assumption, the list is incomplete. The set of real numbers cannot be arranged in a list, and cannot therefore be put into one to one correspondence with the integers. Whatever else it is, the cardinality of the real numbers is not aleph-null.

The attempt to determine the cardinality of the real numbers, and the acute hostility to his work, is what sent Cantor over the edge. Personally, he was convinced the cardinality of the real numbers was the next highest infinity $\aleph_1$ and that it possessed the value

$$\aleph_1 = 2^{\aleph_0}$$

Cantor spent the rest of his life trying to prove this result. Today, this conjecture is known as the continuum hypothesis, since it aspires to count the totality of points comprising a continuous curve. A little disappointingly for some mathematicians, it has now been proved to be …umm … unprovable.

You can accept it as an axiom, and go on to successfully develop a self-consistent mathematics of the infinite, or you can choose an alternative axiom and go on to successfully develop an alternative, self-consistent, mathematics of the infinite.

Historically, the situation is analogous to the famous parallel postulate. Including the parallel postulate as a geometrical axiom permits the successful development of Euclidean geometry. Replacing the parallel postulate by an alternate but equally consistent postulate, on the other hand, allows the successful development of Non-Euclidean geometries known as hyperbolic and elliptical geometry.

There is a general intuition among mathematicians that simple problems should beget simple answers and complex problems, complex answers. Although this is commonly the case in practice, it can be spectacularly wrong on occasion. Some simple problems yield the most unexpected complexity and, conversely, some complex problems can exhibit surprisingly simple solutions.

I'd like to introduce you to a simple problem whose solution is a mathematical creature of unfathomable complexity, a creature of deep infinity which lives and breathes infinity, the Mandelbrot set.

While we were happy to stick to add and subtract, infinity remained comfortingly far away. Not being used to seeing this monster clearly in the distance, makes it hard to recognise close up. Introduce multiply and divide and infinity comes up close. Unlike many of our earlier glimpses of infinity, you can get as close as you like to the Mandelbrot set, although rather frustratingly, you can never actually touch it. The Mandelbrot set is also intriguing for another reason - it is very much the discovery of that finite device, the computer.

Choose a starting point (x_1, y_1) in the plane. From these coordinates, compute a second point (x_2, y_2) using to the formulas

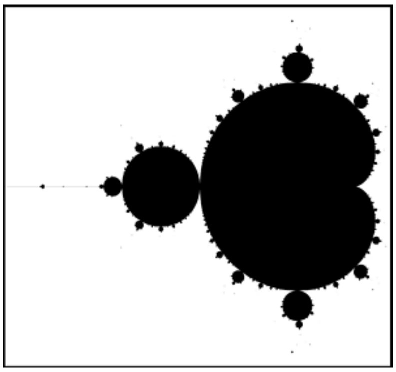
$$x_2 = x_1^2 - y_1^2 + 0.1$$
$$y_2 = 2x_1y_1 + 0.2$$

From the coordinates of the second point (x_2, y_2) compute a third point using the formulas,

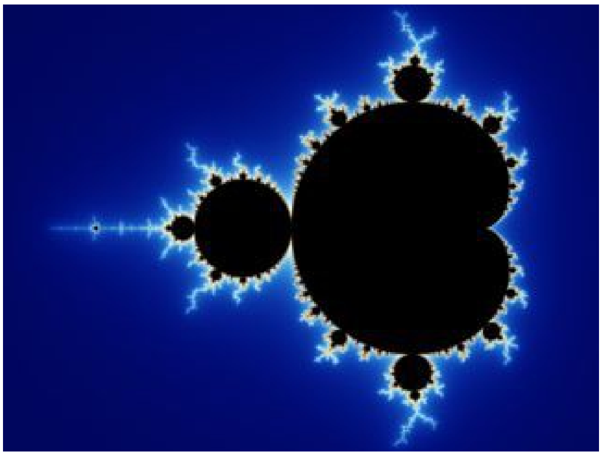
$$x_3 = x_2^2 - y_2^2 + 0.1$$
$$y_3 = 2x_2y_2 + 0.2$$

and so on, obtaining a sequence of points (x_1, y_1), (x_2, y_2), (x_3, y_3) ... With a little spreadsheet experimenting, you quickly discover that, depending on your choice of first point, subsequent points either wander around near the origin indefinitely or accelerate rapidly away. However, it is strangely difficult to predict which sequences will diverge and which will not. A very intriguing image can be obtained as follows.

Select an initial point. If after say 10 iterations, the point has wandered off more than say 1000 units away from the origin, colour the initial point white (depicting unstable). Otherwise colour it black (stable). The choices 10 iterations and 1000 units divergence turn out to be quite arbitrary – similar results are obtained for a large spectrum of such choices.



Now repeat for a neighbouring point, and so on, systematically painting a black and white stability map for an entire region in the (x,y) plane. A simple BASIC program can be written to perform the whole process or you can download one of many such programs from the internet.



The first surprise is the enormity of detail, given the simplicity of the iteration. The detail is extraordinary, and if we ask for improved resolution - a colour contour map of the slide to infinity - there it is, available simply by painting the initial points red, yellow, blue, green … according to whether it takes 10, 20, 30, 40… iterations to travel 1000 units of distance.

http://video.google.com/videoplay?docid=-3638096461401109182

The next surprise is mesmerising. The detail continues down through <u>every</u> scale. The computer image can be zoomed in indefinitely to witness a strange bottomless world of kaleidoscopic symmetry and complexity, regenerating itself on every scale. Fractals, as such objects are now called (since bizarrely, they can be shown to occupy a fractional dimension) are truly monsters from the depths of infinity, up close and personal.

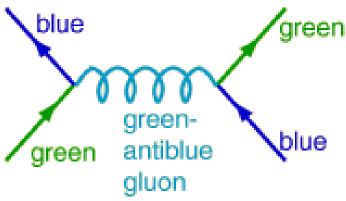
Although it took some time for his work to be accepted by mathematicians, in the end Benoit Mandelbrot fared a little better than Cantor. IBM allowed him to pursue his investigations on fractals while employed in a nominal capacity, before the mathematical community finally accepted fractals as entities worthy of serious attention and Mandelbrot worthy of academic employment.

Today, infinity is no longer the esoteric mathematical abstraction it once was. Infinite processes can and do have finite consequences and important applications. Fractal compression for example, enables massive economies of information storage and transmission, enabling the commercial reality of a host of digital communication devices such a DVD's and mobile phones.

In science, the appearance of infinity in calculations has been traditionally greeted with howls of derision or despair, depending on whether you worked in the experimental or theoretical camps. The onset of infinite values traditionally signalled a flaw in the mathematical modelling of physical reality. Today, modern physics is having to rethink this instinct, as various indications point to the existence of real physical infinities operating at the limits of space, time and matter.

Last year for the first time, physicists at CERN (Geneva) had to seriously consider whether tampering with space at small scales and high energies, might lead to the accidental creation of a microscopic black hole. All theories of such objects involve a singularity of space, the physical realisation of a mathematical infinity turning space inside out and isolating the interior of the black hole from the rest of the universe (but not the rest of the universe from the interior of the black hole).

In theoretical physics, "irreducible" infinities arise in the calculations of modern quantum electrodynamics, the physics of elementary charged particles. Here, infinites are routinely deleted to "enable" a finite answer. Feynman Diagrams constitute a well tested prescription for guiding physicists to which parts of infinity to retain, and which to delete.

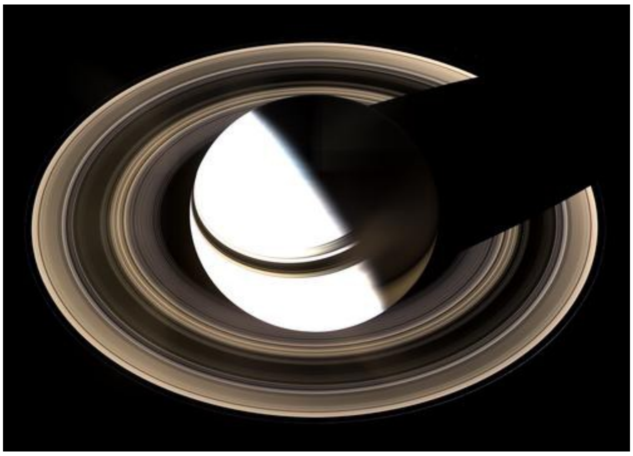


Feynman diagram for an interaction between quarks generated by a gluon.

There's no denying the practical efficacy of the process, some of the theoretical predictions of QED agreeing with measurement to better than ten significant figures. Of course, retaining a finite part of a calculation and ignoring the infinite part, is plain nonsense mathematically. It is also deeply unsatisfying, physically. One can only wonder what wealth of information and understanding of the physical universe is lost by consigning the more substantial part of the answer to the trash can.

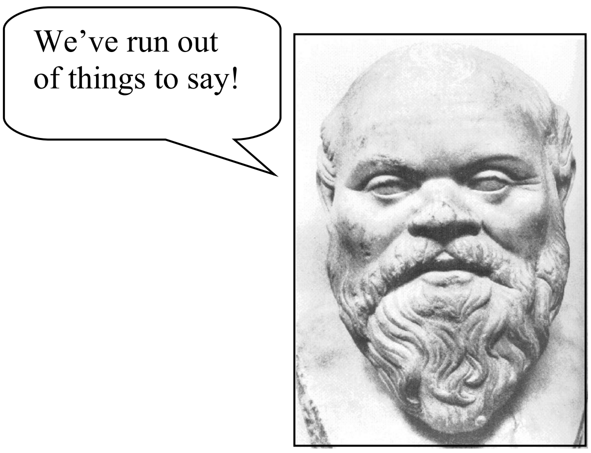
Now it's no longer taboo in science to report glimpses of infinity in the physical world, earlier sightings of the creature are being admitted.

An intractable but little known problem in the theory of Newtonian motion, formerly known as the small divisor problem, has over the past fifty years assumed such notoriety and interest that it is now called Chaos Theory.



Chaos theory predicts the long term unpredictability of evolving real world systems such as: motion of magnets, ring particles of Saturn, Earth-asteroid collisions, stability of the solar system, weather, climate stability, species extinction, stability of financial markets, world peace, etc. The inevitability of small divisors in the governing dynamics ultimately causes all precision of the predicted motion to be lost over time. This phenomenon has become known as the butterfly effect. For planetary orbits, the prediction time is millions of years. For the weather, it can be less than a few days. Perhaps the British obsession for meteorological small talk has some purpose after all.

I'd like to end with a few philosophical speculations. Sometimes we say in desperation, "I don't understand something", when what we really mean is, "I don't believe it". Good tactics in the cut and thrust of everyday life, but it doesn't work too well in mathematics. Like infinity, you might find the following hard to believe but, like Zeno's paradox, that's not the point. The point is that, ultimately, belief plays no role in mathematics, having proved an too unreliable guide in the past. Even its close cousin, intuition, is only allowed to ask questions in mathematics, never answer them.

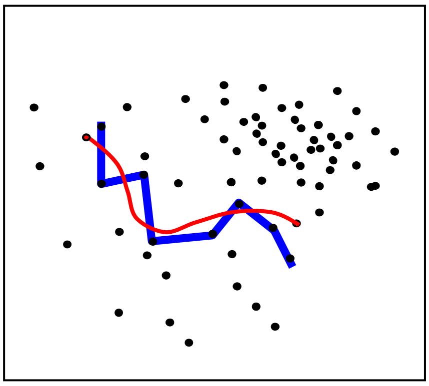


Socrates

The English language, in common with all other human languages, is composed from a finite number of letters. A word is a finite string of letters, so there are only a finite number of distinct words. A sentence is a finite string of words, so there are only a finite number of distinct sentences. Once these are said, nobody can ever say anything new again. Where does that leave human imagination, free will and consciousness?

Philosophers assure us that, without language, we are incapable of thought. A finite language then condemns us to a finite set of possible thoughts. It would seem that in language and thought, we are as limited as that other finite device, the computer. The Turing halting problem of computational mathematics shows us what happens when a machine of finite language and memory, attempts to solve a nontrivial problem. The conclusion is rather depressing. Perfectly straight forward problems exist for which the machine never halts, never reaches a conclusion.

A finite language imposes another critical constraint on human thought. If we map each thought to a single point of a mathematical space, say the plane, then a finite set of thoughts necessarily maps to a finite set of points. If in addition, we arrange for similar thoughts to map to nearby points, we construct a kind of thesaurus of thoughts, or Thought Space.



Thought space

But a finite set of points can never compose a continuous curve segment, however short, so every point in this space is necessarily discrete and isolated from its nearest neighbour by a certain minimum distance $\delta > 0$. In other words, the space of human thoughts is not only finite but quantised, every thought differing from every other thought by a finite amount.

Now we have no reason to suppose that the phenomena of nature pay exact homage to human language and thoughts. Atoms, planets, stars and galaxies existed long before humans arrived on the scene to think about them. So we should not expect an exact coincidence of points representing the phenomena of nature with those representing human thought.

An entity of nature requiring for its understanding two distinct thoughts closer than δ would be inaccessible to human thought, our thought space being insufficiently well populated to resolve the distinct elements. A sequence of thoughts (the blue track), such as we may wish to construct to establish the rigorous proof of some statement, could not therefore describe reality (the red curve) to better than a certain degree of precision. Like our visualisations of the Mandelbrot set, human models of nature must remain necessarily course-grained to some degree, incapable of ultimate clarity. Any natural phenomenon requiring more than a certain precision of thought would be beyond human understanding.

While we have no reason to suppose that the phenomena of nature pay homage to human language and thoughts, we might indeed expect the converse. The evolutionary success of our species in finding and applying thoughts which resonate with reality, should lead us to expect that we steadily populate thought space more densely in regions where greater understanding of reality pays dividends in species survival.

Where human language proves under-populated in describing reality, new words (or more precisely, new thoughts assigned to old words) are introduced. If these thoughts find sufficient resonance with reality, they become established and rapidly propagate, leading to other new words and thoughts, until the once sparse region becomes populated to a density which better describes the real world. This Darwinian-like evolution of the landscape of thought space proves ruthlessly effective in the selection and propagation of successful thoughts.

Over time, this process itself has developed into a highly evolved form of language called mathematics, barely recognisable in either form or purpose with its ancestral origins. Its numbers form the words, its equations the sentences and the laws of mathematics the grammar by which new sentences are constructed and new results and connections discovered. Infinite values and infinite processes arise quite naturally in this *translanguage* and are pivotal in developing tools of ever increasing power and resolution in our quest to understand ourselves and nature.

Rather than evading the infinite, shouldn't we be more concerned with escaping the shackles of the finite? To understand problems like human imagination, free-will and consciousness, perhaps we should be speaking the language of mathematics.

Blake's poem

Eternity in an hour title

(recheck calculations)
Complex numbers – analytic continuation while numbers were 1-dim you had to step over infinity. As soon as you admit 2-dim complex nos. you can walk around infinities
Go to complex numbers, infinities are tamed (you can't walk over infinity but you can walk round one)

Cantor – some biography

Zero and infinity are intimately related, zero as a local portal to infinity
There are different types of zero – linked to different types of infinity
If you admit continuity, as $x \to 0$, $1/x \to \infty$
Time for x to reach 0 is finite, so time for 1/x to reach infinity is finite
Multiply just seems a more efficient way to do adds
Divide the way to undo the multiply. But divide is portal to local infinity
Mathematicians always very anxious to avoid dividing by zero
Problems where this happens implicitly leads to qualitative changes
But can divide by zero without knowing it. (viz andy's 1 = 0 proofs)
Which infinity is 1/0?
p/q lines passing thru origin giving inconsistent values for 0/0
Touchline judges in the game y=1/x
Divergent 1/x and conditionally convergent series
Conditionally convergent series have strange properties
Continued fractions
Conditionally convergent series rearrange to different sums

Need infinity to define probability Googal kolmogorov axioms probability
Probability theory requires infinity for its definitions
20% probability it will rain tomorrow?

But some physical observations demand strange theories.
Physics infinities number of atoms, size of universe, ration of strong force to gravity (see penrose)

Min and max merge leads to catastrophes
When it does happen, unintuitive discontinous things happen
Switching in Algebra and Geometry representations can give catastrophes
Primes lnx/x
Primes 100 consecutive nonprimes
Russel and Whitehead, 2+2 =4 by page 120
Law of the excluded middle.

Negative numbers, imaginary numbers and transfinite numbers “work of devil”
Google “transfinite numbers and god/devil”
Suppose you don’t know rules of chess but watching a game, you gradually start working them out.

For a very modest £20 (check) you may have heard Lord X give his talk on *** last month, so when I was preparing my talk I asked the burser if the school will charge admittance tonight. A little disappointingly, I was told no, and in fact at some stage Senior Management had considered paying a few people to attend, but eventually decided this would set the wrong tone. So I’m very glad to see…

Just in case you wondered: The Language of Thought Hypothesis (LOTH) postulates that thought and thinking take place in a mental language. This language consists of a system of representations that is physically realized in the brain of thinkers and has a combinatorial syntax (and semantics) such that operations on representations are causally sensitive only to the syntactic properties of representations. According to LOTH, thought is, roughly, the tokening of a representation that has a syntactic (constituent) structure with an appropriate semantics. Thinking thus consists in syntactic operations defined over such representations. Most of the arguments for LOTH derive their strength from their ability to explain certain empirical phenomena like productivity and systematicity of thought and thinking.

<u>School:</u>
Google “transfinite numbers and god/devil/physics/psychology/consciousness/freewill/imagination etc.
Google Negatives, imaginarys and transfinites “work of devil”
Google images of tortoises and hares
Google work of devil infinity beast numbers

Sources
Bridges to infinity book
Some big numbers Hardy Littleton, Rumanagan, Grahams number
Googal 10^{100}

Poster

Ornate aleph-null characters
Philosophy Goethe quote maths is identity and picture, Hegel picture quote
Mandelbrot set
Headline “Physicists fear new accelerator will create black hole” CERN pix
Feynman diagram
Socrates “We’ve run out of things to say”
Zeno’s paradox
Transfinite numbers work of the devil
Latest Saturn image
Hitchhikers Guide image
See eternity in an hour…

total of grains is approximately 0.0031% of the number of atoms in 12 grams of carbon-12 and probably more than 200,000 times the estimated number of neuronal connections in the human brain (see large numbers).

Philosophy & infinity

Kant   Luminol & phenomenal

Leibinitz

Plato

Sufi