Dear Professor Leedham-Green, 18th Oct 2015

I got interested in infinite products after reading Kanigel's biography of Ramanujan. Pure maths really isn't my thing and I know practically nothing about infinite products, but from what I've read there seems precious little systematic development of the subject anyway. My 1980 copy of Gradshteyn and Ryzhik's *Table of Integrals, Series and Products* for example lists just eight odd-ball results for infinite products (one of which is Ramanujan's infinitely nested square roots), compared to many thousands of infinite sums: one wonders why the word 'Products' ever appeared in the title.

Anyway I got to wondering if there was some easy way of always generating an 'infinite product' from an 'infinite series'. This would loosely imply there were just as many of each and that in some sense they were just two different ways of looking at the same object.

Suppose we start off with the simple geometric series

$$\left(\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{8}+\frac{1}{16}+\ldots$$

and replace the *sum* of the first two fractions by a *product* of two fractions (the first being unity, say)

$$\left(\frac{1}{1}\times\frac{3}{4}\right)+\frac{1}{8}+\frac{1}{16}+\ldots$$

Then do a similar thing to the first three fractions,

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)+\frac{1}{16}+\ldots$$

replacing them by the product of three fractions:

$$\left(\frac{1}{1}\times\frac{3}{4}\times\frac{7}{6}\right)+\frac{1}{16}+\ldots$$

and so on, always choosing the next fraction so as to make the partial sums and products equate.

We can keep doing this as long as we like, systematically replacing the infinite series term by term with an infinite product. It gets fairly tedious and prone to arithmetic error very quickly and anyway there is a much better way.

Let a_n denote the <u>terms</u> of the infinite sum $S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Let f_n denote the <u>factors</u> of the infinite product $P = f_1 \times f_2 \times f_3 \times \ldots \times f_n \times \ldots$

Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ denote the <u>partial sums</u> of S

Let $p_n = f_1 \times f_2 \times f_3 \times \ldots \times f_n$ denote the <u>partial products</u> of P

Defining the factors f_n by $f_1 = \frac{a_1}{1},\quad f_2 = \frac{a_1+a_2}{a_1},\quad f_3 = \frac{a_1+a_2+a_3}{a_1+a_2},\quad \ldots\quad f_n = \frac{a_1+a_2+a_3+\cdots+a_n}{a_1+a_2+a_3+\cdots+a_{n-1}}$

we see immediately that the partial sums are identical in value to the partial products:

$s_n = p_n$ or more fully,

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \quad = \quad \frac{a_1}{1} \times \frac{a_1+a_2}{a_1} \times \frac{a_1+a_2+a_3}{a_1+a_2} \times \ldots \times \frac{a_1+a_2+a_3+\cdots+a_n}{a_1+a_2+a_3+\cdots+a_{n-1}} \times \ldots \quad (1)$$

Equation (1) gives us a useful method for generating infinite products from infinite series (and vice versa). Intuition suggests the infinite product converges if and only if the infinite series converges.

I don't know if this result is familiar or useful to mathematicians; I don't know if it is meaningful in the most rigorous sense. I do know it looks trivial and obvious and circular, but so do a lot of useful results (Jordan's Curve Theorem for example!)

Anyway, straight off it enables one to generate as many infinite products as there are infinite series, and vice versa.

For the special case of a (convergent) geometric series for example, where we can always evaluate the partial sums, we find

$$\sum_{m=1}^{\infty} r^{m-1} = \frac{1}{1-r} = \prod_{m=1}^{\infty} \frac{1-r^{m+1}}{1-r^m}$$

It is true that substantial cancelling is 'available' on the right hand side, but I simply choose to avoid cancelling: wholesale cancelling generally is likely to prove problematic with infinite products. Do we insist on wholesale 'simplification' of terms in an infinite series? No, in fact we're rather careful to avoid it, knowing that generally 'here be dragons' nearby.

Writing $R = \frac{1}{r}$ for slight numerical convenience, we arrive at a simple generating function for infinite products and their computation

$$\sum_{m=1}^{\infty} r^{m-1} = \frac{R}{R-1} = \prod_{m=1}^{\infty} \frac{R^{m+1}-1}{R^{m+1}-R}$$

For example, taking R = 2 gives

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+ \ldots = \frac{3}{2}\times\frac{7}{6}\times\frac{15}{14}\times\frac{31}{30}\times\frac{63}{62}\times \ldots = 2$$

Taking R = 3 gives

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+ \ldots = \frac{8}{6}\times\frac{26}{24}\times\frac{80}{78}\times\frac{242}{240}\times\frac{728}{726}\times \ldots = \frac{3}{2}$$

Both expressions are simple yet not obvious. As a minimum I would have expected to find a few listed in Gradshteyn and Ryzhik's tables and the internet. But no, I can't find anything like them listed anywhere.

Given convergence, infinite products can then be scalar multiplied, added, subtracted, multiplied and divided, just like ordinary numbers. The factors of two convergent infinite products can be rearranged and combined into pairwise products under multiplication, and into pairwise ratios under division, leading to an ability to combine infinite products (and their corresponding infinite series) just like ordinary numbers.

It therefore seems to me it might prove an enormously useful and powerful tool to be able to transform any given infinite sum into a corresponding infinite product, and vice verse, rather like Fourier Transforms and Laplace Transforms. It might even open up a whole new avenue of attack on Riemann's Conjecture, which rather suggestively, involves both infinite products and infinite sums.

Regards, Godfrey Powell