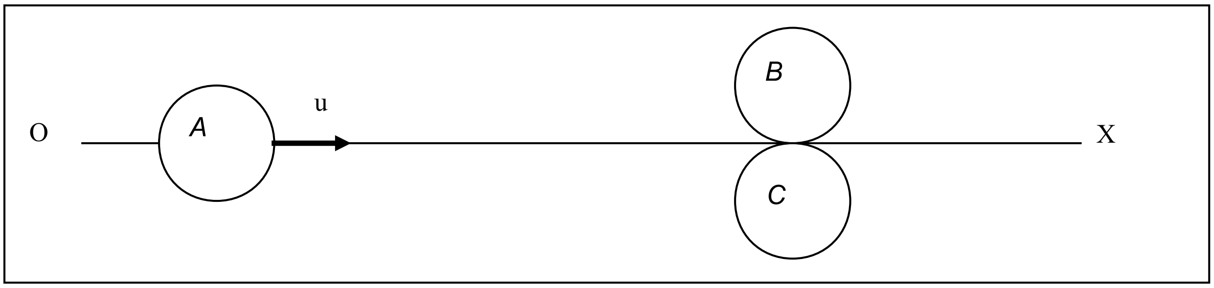
Here be dragons



Hi. That look's painful.

Yes its proving a bit of a problem.

Anyway I can help?

I'm not sure really. You see, Newton's equations of motion are so universally applicable, whether calculating snooker shots, break horse power or trips to Mars, that it comes as a bit of a shock to run into an innocent looking problem that they cannot tackle.

Yes, well I didn't really mean...

Look, line up three identical snooker balls *A*, *B*, *C* with *B* and *C* touching either side of the OX line. Then send *A* in towards them with speed u along OX. The question is what happens?

Well that's easy, they collide, obviously.

Yes. A three body collision, in fact. At sixth form, we only get to solve two body collisions.

But surely its just a case of a bit more algebra?

Well, yes. And no. Intuitively, symmetry suggests that *A* collides simultaneously with *B* and *C*, that *B* and *C* move to the right at equal speeds (because of the symmetry) and diverging from each other along lines 30^{o} from OX, since at the instant of collision, the centres of *A*, *B* and *C* form an equilateral triangle. symmetrically straddling the axis OX.

I'll buy that. So what's the fuss?

Well for example, what are the final speeds and how does *A* move after the collision?

I don't know, but I have a sinking feeling you're going to tell me.

Being a boring mathematician, I first have to make explicit lots of boring assumptions before even thinking about equations.

Oh good. Wife on lates tonight, is she?

All the balls are identical spheres with equal mass, radius and uniform density, so that I can treat them as point particles whenever I want. ***Yeah, Yeah...***

But at the same time, I can treat them as rigid bodies with spatial extension whenever I want. ***Eh...?***

The coefficient of restitution for each collision is 1, so there is no conversion of mechanical energy into heat, sound, vibration etc. during collisions. ***Whose constitution....?***

Also the surface of the balls are dynamically smooth, so that on colliding, the impulse acts exactly through the centre of each ball and does not give rise to any rotational motion. ***Gulp...!***.

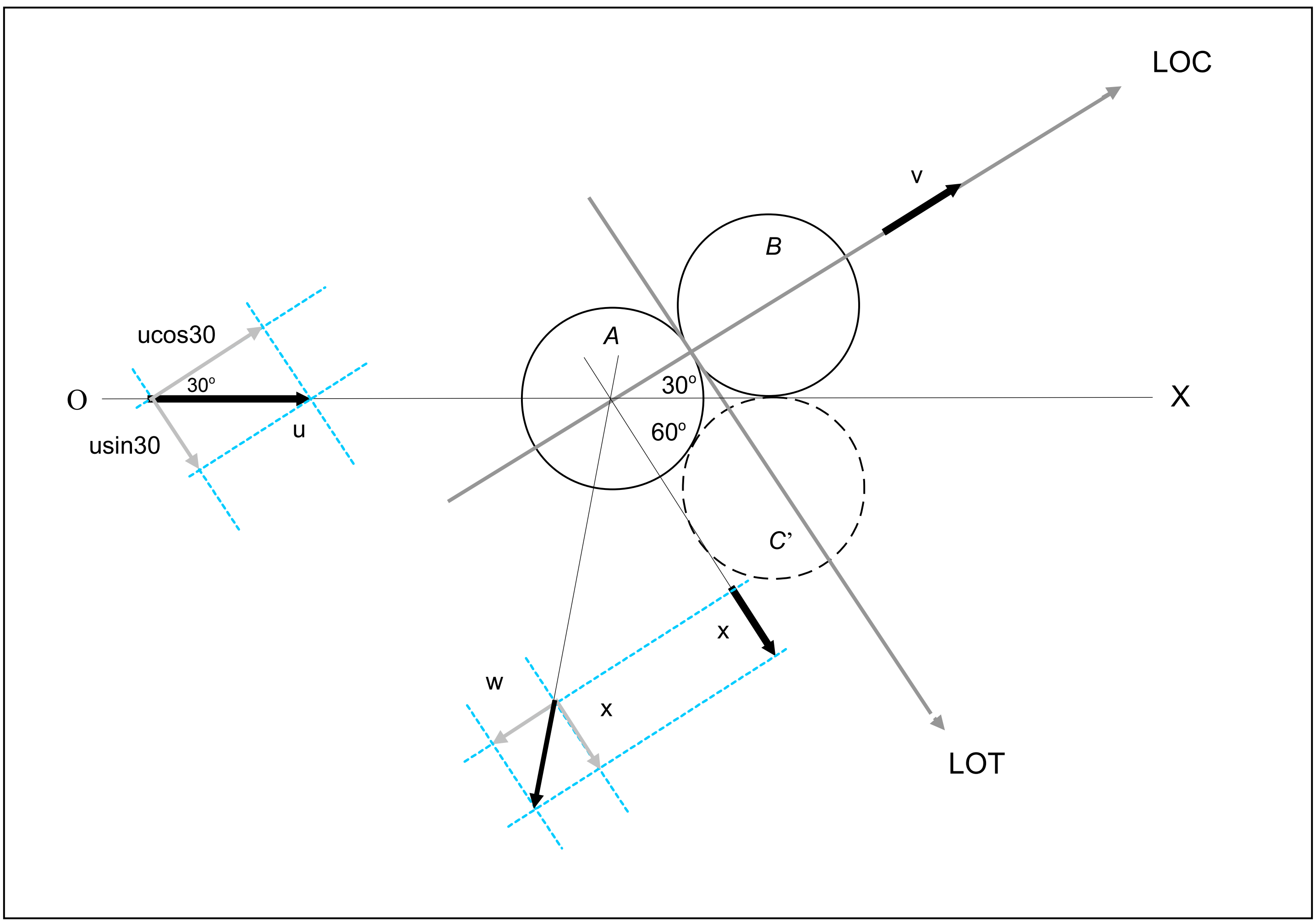
The kinetic energy of motion is purely one of translation with no rotation, that is all the balls slide frictionlessly across the baize without rolling. ***What!!!?***

Now just stop right there...! This is ridiculous! Can't you guys solve anything without making the whole thing trivial and unreal?

Well, no, I guess not. But then, I'm not sure anyone else can. The point is, even with all these simplifying assumptions, the problem is still awkward. Relaxing any of these assumptions just makes it worse. That's the price of certainty.

OK, let's get on with it, I have a life.

*C*onsider what happens if *C* is not present. Let's agree to record its position by a dashed outline *C*', just like in the movies. *A* now collides obliquely with *B*, a straight forward two body collision which we cover in M4. First I'll draw some useful construction lines LOT and LOC (lines of tangents and centres) and choose the positive directions.



Is all that stuff meant to help?

Then I resolve *A*'s initial velocity **u** into two components usin30 and ucos30 (parallel to LOT and LOC respectively), and write down the following equations:

Along LOT: $x = u\sin 30$ (1)

Along LOC: (Momentum) mucos30 = mv - mw (2)

(Restitution) ucos30 = v + w (3)

Solving gives, x = u/2 v = ucos30 = $u3^{1/2}/2$ w = 0.

So *A* suffers a deflection of 60° clockwise and a decrease in speed by a factor x 1/2. *B* moves off along a line 30° anticlockwise from OX with a speed $u3^{1/2}/2$. The final state is summarised in Fig.3A(ii)

So why initially draw A as being deflected by much more?

Because when I drew the diagram, I hadn't yet done the calculation. Now I have, I can draw on the exact angle. OK?

We have solved the two body collision *A* ⮝*B*. *B* moves off and clears *C*'. Not so *A*. The centre of *A* actually moves along a line parallel to LOT, as it happens, tangential to the circle *C*'. Clearly *A* would collide obliquely with *C* if *C* were centred anywhere along the line LOT.

Suppose now, instead of actually removing *C* from the scene, we just positioned it ever so slightly away from the *A* ⮝*B* impact, say a little along the line LOT. In maths, a miss is as good as a mile, so as far as the *A* ⮝*B* collision is concerned, *C* is just a bystander and all the above analysis holds good.

Now clearly, in a very short time, *A* will collide with *C*. But by then *B* is clear and we will again have a fully specified two body collision, this time *A* ⮝*C*, with the initial positions and velocities of *A* and *C* fully specified. In fact we see that we have virtually the same collision dynamics as before, if from above, we twist our viewing angle through 60° clockwise and take account of *A*'s reduced speed, u/2. So we don't have to repeat the analysis to find out what happens, although we can if you prefer.

Er, is that the time?

Anyway, *A* suffers another deflection of 60° clockwise and another decrease in speed by a factor x 1/2. *A* therefore finally moves away from the collision area along a line 120° clockwise from its original direction of motion with a final speed u/4

B moves along a line 30° anticlockwise from OX with speed $u3^{1/2}/2$.

C moves along a line 30° clockwise from OX with speed u = $u3^{1/2}/4$. The final motions are summarised in Figure 3A(iii).

Is that it? Not as symmetric as I expected.

You're right, its not. And its here that a dreadful dark secret of this problem begins to emerge. Consider, almost as an afterthought, what would happen if instead of moving *C* slightly back, we moved *B* instead and allow *A* ⮝*C* first. After all, in the real world we cannot expect to position anything with arbitrary accuracy.

Oh no, I'm not going through that lot again, good night!

No, no, wait. We don't have to! Just flip the picture over and look at it from the back. Now the top ball is a little further back. Obviously, the mathematical analysis doesn't change just because I've flipped the picture over, so the reversed view tells us what would happen if *A* collided with *C* first. Look, I've drawn out both outcomes side by side in Figure 3.

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Fig. 3A(i)

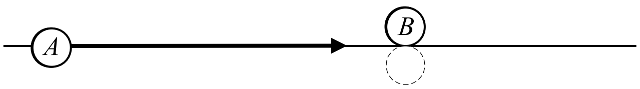


Fig. 3A(ii)

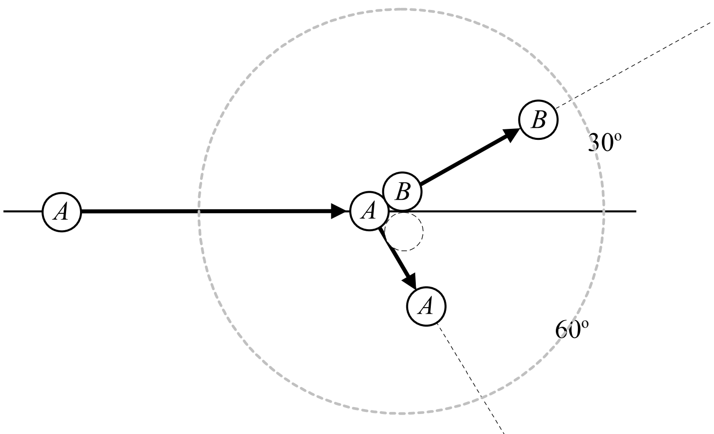


Fig. 3A(iii)

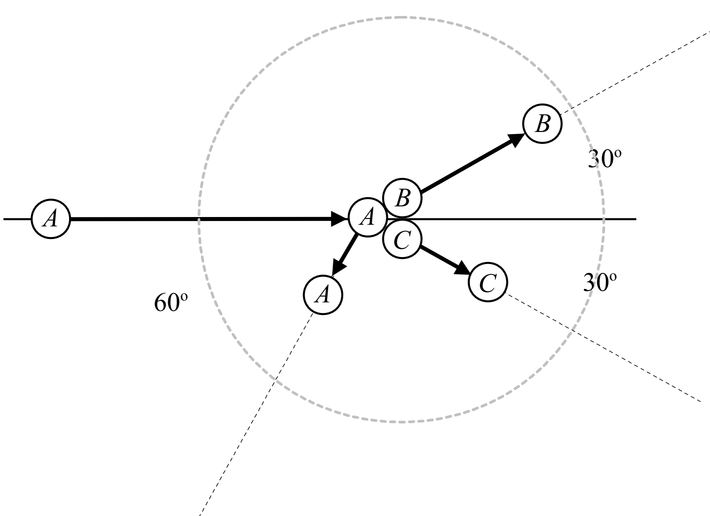


Fig. 3B(i)

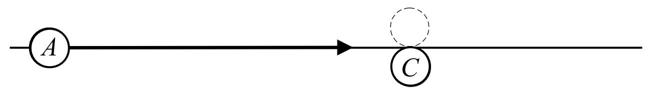


Fig. 3B(ii)

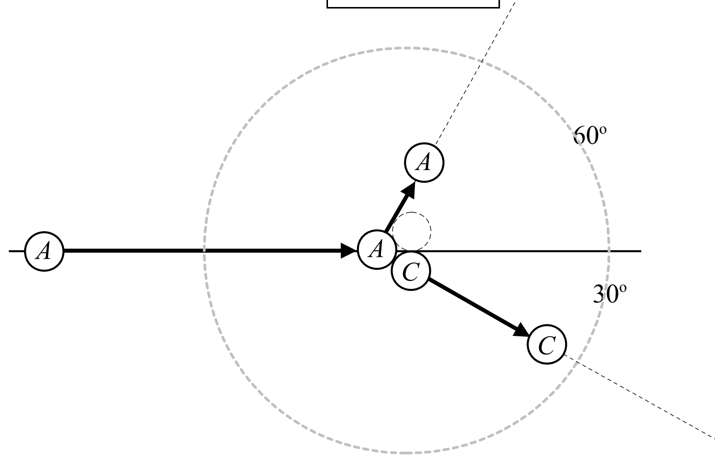
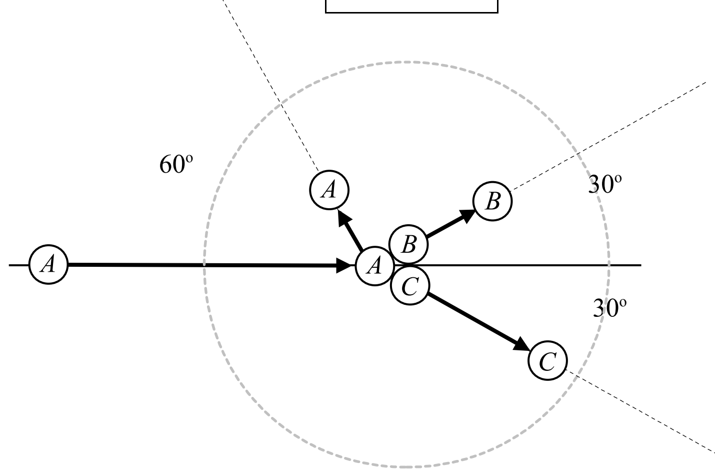


Fig. 3B(iii)



While we can't expect infinite precision in the real world, we should expect that a small change in the initial conditions causes only a correspondingly small change in the final motions. Without this concept of structural stability, mathematical models would be unable to comment on anything in the real world.

Unfortunately, we now see that just such a small change in initial conditions, results in a gross change in the final motions. Depending on the exact position of *B* and *C*, and hence the order of impacts, *A* is deflected through 120° clockwise if *A* ⮝ *B* first, or 120° anticlockwise if *A* ⮝ *C* first. Other asymmetries in the speeds of *B* and *C* (though not their directions) are also evident.

But this just doesn't make sense. You're saying that however carefully we set up the experiment, we could never predict which of two completely different outcomes will actually happen. That's crazy talk!

Yes its all very unsatisfactory.

So where did we go wrong?

At the beginning.

Does your wife ever threaten you, physically, I mean?

Look, remember when you got all huffy about those assumptions? Well, we should have persevered. One or more of those assumptions were mutually inconsistent with the others. It's not that Newton's equations fail for this problem. This is not a real-world problem. You cannot actually set it up in the real world under those precise assumptions.

What are you talking about? They weren't my idea in the first place!

No, I know. But in the real world we know there is no such thing as a perfectly rigid body. And there's no such thing as two events happening simultaneously either. They are just mathematical abstractions which usually make a model easier to solve, an analytic convenience. If you push the abstractions too hard, then sometimes, in certain critical problems, you run into fatal inconsistencies. Here be dragons, my friend.

But where be dragons, be also buried treasure, maybe. Anyway I can set up the real world problem! In fact, that's all I wanted to do in the first place. Come on, grab a beer. Let's up to the snooker room and see what actually happens. You got me at this nonsense now. And for God's sake, leave that bloody pen behind! This time, we'll do it my way.
